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Comparison of information structures and completely positive maps

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Abstract

A theorem of Blackwell about comparison between information structures in classical statistics is given as an analogue in the quantum probabilistic set-up. The theorem provides an operational interpretation for trace-preserving completely positive maps, which are the natural quantum analogue of classical stochastic maps. The proof of the theorem relies on the separation theorem for convex sets and on quantum teleportation.

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1. Introduction

Consider an observer with access to a quantum particle S which is entangled with another quantum particle N. Let $\mathcal{H}_S$ and $\mathcal{H}_N$ be the corresponding Hilbert spaces and Φ the density operator over $\mathcal{H}_N \otimes \mathcal{H}_S$ that represents the state of the bipartite system. This paper considers the information that the observer can garner about N via measurements over S. We call the triple $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ an information structure.

The approach follows Blackwell's similar analysis in classical statistics [3]. The analysis is comparative. Given a pair of information structures, $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ and $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$, we seek for a condition under which $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ can be said to be more informative than $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$. The concept of being 'more informative' is understood operationally, that is in terms of the payoffs that the observer can expect in a certain class of games. We consider the two scenarios that correspond to these structures: in the first scenario, the observer has access to a particle S such that the joint state of N and S is Φ; and in the second scenario, he has access to a particle T such that the joint state of N and T is Ψ. In both scenarios, the observer is assumed to be engaged in some decision problem, or game, in which he has to choose an action depending on his estimation about the outcome of future measurements over N. We say that the information structure $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ is better than the information structure $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$ if, for every possible game, the expected payoff for the observer in the first scenario is at least as good as the expected payoff in the second scenario.

Before delving into formal definitions, consider the following example. Assume that all particles have spin-$\frac{1}{2}$ and that

$$\Phi = \tfrac{1}{2}(|01\rangle\langle 01| - |10\rangle\langle 01| - |01\rangle\langle 10| + |10\rangle\langle 10|)$$

is the density matrix of a singlet and that

$$\Psi = \tfrac{1}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)$$

is the density matrix of two independent random states. Suppose, for example, that the observer is involved in the following game. After performing measurements over his particle, he has to guess the component of the spin of N along axis $\hat{n}$ for some fixed unit vector $\hat{n} \in \mathbf{R}^3$. His payoff is +1 for a correct guess and −1 for a wrong guess. In the first scenario, when the observer has access to a particle S such that the composite system of N and S is at state Φ, he can guarantee payoff +1 by measuring the spin of S along $\hat{n}$ (which is, with probability 1, in the opposite direction to the spin of N). On the other hand, in the second scenario, when the observer has access to a particle T such that the composite system of N and T is at state Ψ, measuring T will give him no help, and whatever strategy he uses for his guess, his expected payoff is zero. In fact, the situation described by Φ is better than the situation described by Ψ, in the sense that in *every* 'game' of this type—a formal definition of game is given below—the observer can do better (in a weak sense) in the former situation.

The advantage of Φ over Ψ is also reflected in the fact that an observer who has an access to a particle S such that the composite system of N and S is at state Φ, can perform physical manipulations on S that transform the state of the bipartite system to Ψ. To do that he applies S to the completely depolarizing map given by

$$\rho \mapsto \frac{1}{4}\sum_{\mu=0}^{3} \sigma_\mu \rho \sigma_\mu, \tag{1}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Essentially, theorem 2 says that if $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ is better than $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$ in the sense that for all possible games it is better for the observer to play the game in the scenario corresponding to $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$, then the observer can perform physical manipulations on S that transform Φ into Ψ.

As a second example, consider a pair of spin-$\frac{1}{2}$ particles N and Q whose joint state is given by

$$\Upsilon = \Upsilon_{NQ} = \tfrac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|).$$

This is a separable state, corresponding to a mixing of two pure product states. Assume that the observer, who as before has an access to the second particle Q, has to guess the spin of the first particle N along axis $\hat{n}$ for some unit vector $\hat{n}$, with payoffs as before. If $\hat{n} = \hat{z}$ (the z-axis) then the observer can guarantee payoff +1 by measuring the spin of Q along $\hat{z}$ (which is, with probability 1, in the opposite direction to the spin of N along the same axis). Thus, in terms of the operational approach taken by this paper for measuring of information, Υ is strictly better than Ψ. However, if $\hat{n} = \hat{x}$ (the x-axis), one can verify that whatever strategy the observer employs, his expected payoff under Υ would be zero. Therefore, Φ is better than Υ. Note that the observer can transform a pair of particles at state Φ to a pair of particles at state Υ by applying over S the map

$$\rho \mapsto \tfrac{1}{2}\sigma_0\rho\sigma_0 + \tfrac{1}{2}\sigma_3\rho\sigma_3,$$

where σ_μ are given in (2), and he can transform a pair of particles at state Υ to Ψ by applying over Q the completely depolarizing map (1).

Note that in the last example, although Υ and Ψ are both separable, Υ is in some games strictly better than Ψ and is always at least as good as Ψ. Thus, the operational comparison of information structures is not just a comparison of the amount of entanglement between the two parts. Indeed, quantum particles can be correlated without being entangled [4], and this correlation should also be taken into account when comparing information structures.

Finally, we remark that the order 'better' over information structures is a partial order. There exist pairs of structures Φ and Ψ that are incomparable, that is for some games Φ can yield strictly higher payoffs than Ψ and for some games Ψ can yield strictly higher payoffs than Φ.

Section 2 introduces Blackwell's theorem in classical statistics. The quantum analogue is given in section 3 and proved in section 4. Section 5 discusses relationship to quantitative measure of correlation and section 6 discusses the difference between the classical and quantum set-up, which is related to the existence of positive maps which are not completely positive.

2. Blackwell's theorem in classical statistics

A *classical information structure* is given by a triple (N, S, p), where N and S are finite sets and p is a (classical) distribution over $N \times S$, i.e. $p = (p_{\{n,s\}})_{n\in N, s\in S}$ such that $p_{n,s} \geqslant 0$ and $\sum_{n,s} p_{n,s} = 1$. We can think of N and S as the sets of possible states of two classical particles.

A *game* is given by a finite set A whose elements are called *actions*, each action $a \in A$ corresponding to a payoff function $M^a : N \to \mathbf{R}$. The game is played as follows: first a pair $(n, s) \in N \times S$ is randomly chosen according to p. The observer sees s (the state of particle S) and then chooses an action $a \in A$. The observer's payoff is given by $M^a(n)$. We assume that the observer is *rational*, i.e. that he chooses his action according to a *strategy* that maximizes his expected payoff. Formally, a strategy is given by a partition $\mathcal{P} = \{P^a\}_{a\in A}$ of the set of signals (that is, P^a are disjoint subsets of S whose union is S). If the observer uses that strategy, then, if he sees the signal s, he chooses the action a such that $s \in P^a$. His payoff is given by

$$\sum_a \sum_n \sum_{s\in P^a} p_{n,s} M^a(n).$$

A rational player chooses a strategy that maximizes this entity. His expected payoff is thus given by

$$\max_{\mathcal{P}} \sum_a \sum_n \sum_{s\in P^a} p_{n,s} M^a(n),$$

where the maximum ranges over all partitions $\mathcal{P} = \{P^a\}_{a\in A}$ of S.

Let (N, S, p) and (N, T, q) be two information structures. (N, S, p) is said to be *better* than (N, T, q) if, for every game (that is, for every finite set A and for every payoff functions $\{M^a : N \to \mathbf{R}\}_{a\in A})$ the expected payoff to the rational observer if the game is played over (N, S, p) is at least as good as the expected payoff if the game is played over (N, T, q). Thus, the partial order 'better' over information structures is defined in terms of games. Blackwell's theorem ([3], see also [5] for a recent survey) characterizes the same order in purely probabilistic terms.

Theorem 1. *Let (N, S, p) and (N, T, q) be two classical information structures. Then (N, S, p) is better than (N, T, q) if and only if there exists a matrix $F = (f_{s,t})_{s\in S, t\in T}$ such that $f_{s,t} \geqslant 0$ and $\sum_t f_{s,t} = 1$ for every $s \in S$ (i.e. F is a stochastic matrix) and*

$$q_{n,t} = \sum_s p_{n,s} f_{s,t} \qquad \text{for every } n, t. \tag{3}$$

Note that every stochastic matrix $F = (f_{s,t})_{s\in S,t\in T}$ corresponds to a linear transformation $\mathcal{F} : \mathbf{R}^S \to \mathbf{R}^T$ that transforms probability distributions over S to probability distributions over T. We call $\mathcal{F}$ a (classical) stochastic map. If we think of p and q as elements of $\mathbf{R}^{N\times S}$ and $\mathbf{R}^{N\times T}$ then using the natural isomorphisms $\mathbf{R}^{N\times S} \leftrightarrow \mathbf{R}^N \otimes \mathbf{R}^S$ and $\mathbf{R}^{N\times T} \leftrightarrow \mathbf{R}^N \otimes \mathbf{R}^T$ (3) can be written equivalently as

$$q = (\mathcal{I} \otimes \mathcal{F})p, \tag{4}$$

where $\mathcal{I}$ stands for the identity map $\mathcal{I} : \mathbf{R}^N \to \mathbf{R}^N$.

In particular, a necessary condition for (N, S, p) to be better than (N, T, q) is that p and q induce the same marginal distributions over N. For this reason, classical accounts of Blackwell's theorem usually define information structures using the conditional distribution of s given n, and not in terms of the joint distribution as above. The two formulations are equivalent since, in classical probability, the joint distribution is uniquely determined by the marginal distribution of n and the conditional distribution of s given n.

In the statistical literature, the set S is viewed as a set of possible *signals* to the observer. The application of the stochastic map $\mathcal{F}$ in (4) is interpreted as *simulation*. If the observer receives a signal s he creates, or simulates, a new signal t from the set T, distributed according to the sth line of the matrix F. If the distribution of (n, s) was p, the simulation process results in a new signal t such that the joint distribution of (n, t) is q. Since stochastic maps correspond to all the physical manipulation that can be performed over classical particles, the physical meaning of the application of $\mathcal{I} \otimes \mathcal{F}$ over p is that during the simulation process the observer performs manipulations only upon his part of the bipartite system.

3. Blackwell's theorem in quantum statistics

In the quantum probabilistic set-up, an information structure is given by a triple $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$, where $\mathcal{H}_N$ and $\mathcal{H}_S$ are two finite-dimensional Hilbert Spaces and $\Phi = \Phi_{NS}$ is a density operator over $\mathcal{H}_N \otimes \mathcal{H}_S$, representing the state of a bipartite system of two particles N and S, of which the observer can only access S. Slightly abusing notation, we sometimes refer to the state Φ as the information structure in cases where there should be no confusion to which particle in a pair of particles at state Φ the observer has access.

There are two concepts which need clarification before we can formulate an analogue of theorem 1 in quantum probability. First, we have to define the notion of game. Second, we have to find the appropriate analogue of stochastic maps.

We start with the second task. As mentioned above, a stochastic matrix corresponds to a linear mapping from classical probability distributions over one set to classical probability distributions over another set. The first quantum analogue that comes to the mind is a linear mapping that transforms density operators into density operators. These are sometimes called positive maps. But there is a crucial difference between stochastic maps in classical statistics and positive maps in quantum statistics. Whereas for every classical stochastic map $\mathcal{F}$, $\mathcal{I} \otimes \mathcal{F}$ is also stochastic, there exist positive maps $\mathcal{E}$ such that $\mathcal{I} \otimes \mathcal{E}$ is not positive (a well-known example is given by the transpose map). We say that $\mathcal{E}$ is completely positive, if $\mathcal{I} \otimes \mathcal{E}$ is positive over $\mathcal{H}' \otimes \mathcal{H}_S$ for every $\mathcal{H}'$, where $\mathcal{I}$ is the identity map over $\mathcal{H}'$. For more information about completely positive maps see, for example, [7].

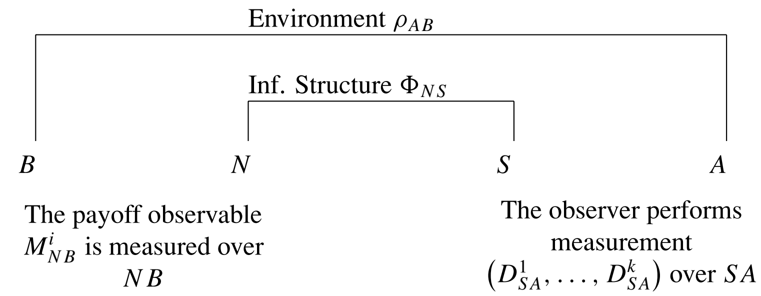
We now turn to the definition of game in the quantum framework. Consider again the spin guessing the game described in the introduction. It can be thought of as a game with two actions, namely 'guess up' and 'guess down'. The observer, after performing measurements over his particle, has to choose one of these actions. If he chooses the first action, his payoff is +1 if the spin of N is up, and -1 otherwise, i.e. the payoff is given by the observable $\hat{n} \cdot \vec{\sigma}$

measured over N, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. If, on the other hand, he chooses the second action (guess down) his payoff is given by the observable $-\hat{n} \cdot \vec{\sigma}$.

Roughly speaking, a *game* is given by a finite set of actions, each action corresponding to some observable that determines the observer's payoff should he choose that action. But in order to achieve the desired result, we need a more general setting, in which some auxiliary bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ at a *fixed* state ρ_{AB} is introduced as a part of the game. The observer can perform his measurements on S and A, and the payoff is determined by observables over N and B. We call the system $\mathcal{H}_A \otimes \mathcal{H}_B$ as the *environment*.

By introducing the environment, we expand the set of games we look at. Game theoretically speaking the environment has a natural interpretation. It represents random entities that, although independent of the information structure, can affect the observer's payoff. In classical statistics, limiting the set of games to games without environment (as we did in section 2) bears no consequences with regard to comparison of information structures. In quantum statistics, however, one must look at the larger set of games with the environment in order to get a concept of comparison which is physically meaningful. The issue is related to the existence of positive maps which are not completely positive, which have no analogue in classical statistics. We return to it in section 6.

Moving now to formal definitions, let $(\mathcal{H}_N, \mathcal{H}_S, \Phi_{NS})$ be an information structure. A *game* over this structure is given by $(\mathcal{H}_A, \mathcal{H}_B, \rho_{AB}, M^1, \ldots, M^k)$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert spaces, ρ_{AB} is a density operator over $\mathcal{H}_A \otimes \mathcal{H}_B$ and $M^i = M^i_{NB}$ is an Hermitian operator over $\mathcal{H}_N \otimes \mathcal{H}_B$ for each i $(1 \leqslant i \leqslant k)$. $M^1, \ldots, M^k$ are called *actions*. Thus, the actions in the game correspond to observables over $\mathcal{H}_N \otimes \mathcal{H}_B$. The game is played as follows: first, the environment $\mathcal{H}_A \otimes \mathcal{H}_B$ is prepared at state ρ_{AB} independently of the system $\mathcal{H}_N \otimes \mathcal{H}_S$. Then the observer can perform measurements on the particles S and A. Using the information he gathered, he chooses one action from the set of available actions $\{M^1, \ldots, M^k\}$. The observable corresponding to that action is then measured, and the numerical outcome of this measurement is the payoff to the observer in the game. Note that the observables M^i need not commute, since only one of them is actually measured. We assume that the observer is *rational*, i.e, that he chooses his action using a strategy that maximizes his expected payoff in the game. The following figure illustrates the role of the particles that are involved in the game.



Note that a strategy in the quantum framework involves the concept of measurement, which does not appear in the classical framework, whereas the classical observer chooses his action given the exact state of his particle, the quantum observer must first choose how to measure his particle and only then he chooses an action, given the outcome of the measurement. Formally, a *strategy* is given by a POVM measurement [7] over $\mathcal{H}_S \otimes \mathcal{H}_A$, i.e. a k-tuple $(D^1, \ldots, D^k) = \left(D^1_{SA}, \ldots, D^k_{SA}\right)$ of non-negative operators over $\mathcal{H}_S \otimes \mathcal{H}_A$ such that $D^1 + \cdots + D^k = I$ (where I is the identity operator). If the observer uses this strategy, he performs this measurement and chooses action M^i if the outcome is i. His expected payoff is given by

$$\sum_{i=1}^{k} \operatorname{tr}\big((\Phi_{NS} \otimes \rho_{AB}) \cdot \big(M^i_{NB} \otimes D^i_{SA}\big)\big).$$

We denote by $R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k)$ the payoff to the observer under the best strategy:

$$R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k) = \max_{(D^1,\dots,D^k)} \sum_{i=1}^{k} \operatorname{tr}((\Phi_{NS} \otimes \rho_{AB}) \cdot (D^i \otimes M^i)), \tag{5}$$

where the maximum ranges over all strategies $(D^1, \dots, D^k)$ (that is, over all k-tuples $(D^1, \dots, D^k)$ of non-negative operators over $\mathcal{H}_S \otimes \mathcal{H}_A$ such that $D^1 + \cdots + D^k = I$).

We now turn to the comparison of two information structures. Let Φ_{NS} and Ψ_{NT} be two information structures. Φ_{NS} is *better* than Ψ_{NT} if, for every bipartite system ρ_{AB} and every set $\{M^1, \dots, M^k\}$ of Hermitian operators over $\mathcal{H}_N \otimes \mathcal{H}_B$ one has

$$R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k) \geqslant R(\Psi_{NT}; \rho_{AB}, M^1, \dots, M^k),$$

that is, the observer can gain in the situation corresponding to Φ_{NS} at least as much as he can gain in the situation corresponding to Ψ_{NT}. We prove the following theorem:

Theorem 2. *Let $\Phi = \Phi_{NS}$ and $\Psi = \Psi_{NT}$ be two information structures. Then Φ is better than Ψ if and only if there exists a completely positive trace-preserving map $\mathcal{E}_S$ acting on S such that*

$$\Psi_{NT} = (\mathcal{I}_N \otimes \mathcal{E}_S)\Phi_{NS}, \tag{6}$$

where $\mathcal{I}_N$ is the identity operation over N.

Note that completely positive trace-preserving maps represent the physical manipulations that the observer can perform on particle S. Thus, the theorem states that Φ is better than Ψ if and only if the observer, starting from a pair of particles at state Φ, can *simulate* a pair of particles at state Ψ by manipulating only his particle. The 'if' part of the theorem is thus intuitively clear (and easily proved, see section 4): if the observer can achieve some payoff r in the situation corresponding to Ψ, then he can achieve the same payoff in the situation corresponding to Φ. To do that he first simulates the situation Ψ by manipulating S and then applies the strategy that achieves r in situation Ψ. The 'only if' part of the theorem says that existence of trace-preserving completely positive maps that transform Φ to Ψ is necessary for the information structure Φ to be better than Ψ in the operational sense of allowing higher payoffs in games.

In particular, it follows from theorem 2 that a necessary condition for Φ to be better than Ψ is that $\operatorname{tr}_S[\Phi] = \operatorname{tr}_T[\Psi]$, where $\operatorname{tr}_S, \operatorname{tr}_T$ are the partial traces over S and T, respectively. This means that the partial state of N is the same in both structures. As remarked in section 2, there is a similar necessary condition in the classical framework.

Theorem 2 characterizes the order 'better' over information structures (essentially states) as the order induced by the set of completely positive, state-preserving maps. See Buscemi *et al* [2] for a related partial order 'cleaner' among POVMs, which is induced by completely positive, identity-preserving maps.

4. Proof of theorem 2

Before proving the theorem, we give another representation of strategies which is more convenient for the proof. Consider a strategy in a k-action game given by a POVM measurement $(D^1, \dots, D^k)$. For a density operator x over $\mathcal{H}_S \otimes \mathcal{H}_A$, the probability of getting outcome i if we perform the measurement $(D^1, \dots, D^k)$ on a particle at state x is given by

$$\delta^i(x) = \operatorname{tr}(x \cdot D^i). \tag{7}$$

Note that δ^i is a completely positive map from Hermitian operators over $\mathcal{H}_S \otimes \mathcal{H}_A$ to 1×1 matrices. Let $\delta(x) = (\delta^1(x), \dots, \delta^k(x))$. Then, for every density operator $x, \delta(x)$ is an element of the simplex $\Delta_k = \{(p_1, \dots, p_k) | p_i \geqslant 0, p_1 + \cdots + p_k = 1\}$ of (classical) probability distributions over the set of actions[1]. Thus, every strategy gives rise to a linear function from density operators to Δ_k. The converse is also true, that is for every linear function δ that attaches an element in Δ_k for every density operator over $\mathcal{H}_S \otimes \mathcal{H}_A$ there corresponds a POVM measurement $(D^1, \dots, D^k)$ such that (7) is satisfied.

Returning to the game defined by ρ_{AB} and actions $M^1, \dots, M^k$, recall that the payoff to the observer if he uses strategy $(D^1, \dots, D^k)$ is $\sum_{i=1}^k \operatorname{tr}((\Phi_{NS} \otimes \rho_{AB}) \cdot (M^i \otimes D^i))$. Written in terms of δ^i this amount is given by $\sum_{i=1}^k \operatorname{tr}\big(\big(\mathcal{I}_{NB} \otimes \delta^i_{SA}\big)(\Phi_{NS} \otimes \rho_{AB}) \cdot M^i\big)$, where $\mathcal{I}$ denotes the identity map and subscripts of maps correspond to the particle over which they act. Thus, $\big(\mathcal{I}_{NB} \otimes \delta^i_{SA}\big)(\Phi_{NS} \otimes \rho_{AB})$ is the density operator over $\mathcal{H}_N \otimes \mathcal{H}_B$ that represents the joint state of particles N and B after the measurement, if the outcome was i, multiplied by the probability to get outcome i. The maximal possible payoff is given by

$$R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k) = \max_{\delta} \sum_{i=1}^{k} \operatorname{tr}\big(\big(\mathcal{I}_{NB} \otimes \delta^i_{SA}\big)(\Phi_{NS} \otimes \rho_{AB}) \cdot M^i\big), \tag{8}$$

where the maximum ranges over all strategies $\delta = (\delta^1, \dots, \delta^n)$.

We now use this notation to prove the easy 'if' part of theorem 2. Let Φ_{NS} and Ψ_{NT} be two information structures. Assume that there exists a completely positive trace-preserving map $\mathcal{E}_S$ acting on S such that $\Psi_{NT} = (\mathcal{I}_N \otimes \mathcal{E}_S)\Phi_{NS}$. We prove that Φ_{NS} is better than Ψ_{NT}. Indeed, let $\delta = (\delta^1, \dots, \delta^k)$ be a strategy in the k-action game defined by the information structure Ψ_{NT}. Thus, δ is a linear function from density operators over $\mathcal{H}_T \otimes \mathcal{H}_A$ to Δ_k. Let $\tilde{\delta}$ be the map-composition of $\mathcal{E}_S \otimes \mathcal{I}_A$ and δ: $\tilde{\delta} = \delta \circ (\mathcal{E}_S \otimes \mathcal{I}_A)$. Since $\mathcal{E}_S$ is trace preserving and completely positive, it follows that $\mathcal{E}_S \otimes \mathcal{I}_B$ is trace preserving and positive, and therefore $\tilde{\delta} = (\tilde{\delta}^1, \dots, \tilde{\delta}^k)$ is a strategy in a k-action game defined by the information structure Φ_{NS}.

Now for every environment ρ_{AB} and Hermitian operators $M^1, \dots, M^k$ over $\mathcal{H}_N \otimes \mathcal{H}_B$, since $\Psi_{NT} = (\mathcal{I}_N \otimes \mathcal{E}_S)\Phi_{NS}$, it follows that

$$\big(\mathcal{I}_{NB} \otimes \tilde{\delta}^i_{SA}\big)(\Phi_{NS} \otimes \rho_{AB}) = \big(\mathcal{I}_{NB} \otimes \delta^i_{TA}\big)(\Psi_{NT} \otimes \rho_{AB}). \tag{9}$$

In particular, it follows from (9) and (8) that

$$\begin{aligned} &\sum_{i=1}^{k} \operatorname{tr}\big(\big(\mathcal{I}_{NB} \otimes \delta^i_{TA}\big)(\Psi_{NT} \otimes \rho_{AB}) \cdot M^i\big) \\ &\quad = \sum_{i=1}^{k} \operatorname{tr}\big(\big(\mathcal{I}_{NB} \otimes \tilde{\delta}^i_{SA}\big)(\Phi_{NS} \otimes \rho_{AB}) \cdot M^i\big) \leqslant R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k). \end{aligned} \tag{10}$$

And since this is true for every δ, if follows from (8) that

$$R(\Psi_{NT}; \rho_{AB}, M^1, \dots, M^k) \leqslant R(\Phi_{NS}; \rho_{AB}, M^1, \dots, M^k). \tag{11}$$

[1] In a game theoretic literature elements of the simplex Δ_k are called *mixed actions*.

Remark 3. Note that in the proof, the only place we used the fact that $\mathcal{E}_S$ is completely positive (and not just positive) is to ensure that $\mathcal{E}_S \otimes \mathcal{I}_A$ is positive. In particular, if there exist a positive trace-preserving map $\mathcal{E}_S$ such that $\Psi = (\mathcal{I}_N \otimes \mathcal{E}_S)\Phi$ then (11) is still satisfied for every game with trivial environment (i.e., games in which $\dim(\mathcal{H}_A) = 1$). We return to this point in section 6.

Turning to the second ('only if') part of theorem 2, let $\Phi = \Phi_{NS}$ and $\Psi = \Psi_{NT}$ be two states such that Φ is better than Ψ. We construct a completely positive trace-preserving map $\mathcal{E}_S$ acting on S such that $\Psi_{NT} = (\mathcal{I}_N \otimes \mathcal{E}_S)\Phi_{NS}$. The main idea is that in order to create T from S, the observer invokes a pair of fictitious agents, Alice and Bob. Alice has access to S and she wants to send Bob the information encoded in the state of system T. To achieve this, they carry the standard teleportation protocol, only that Alice, instead of measuring the system T, performs an alternative measurement on S with the same effect. The existence of such an alternative measurement follows from the fact that Φ achieves higher payoff than Ψ in every game, and, in particular, in games whose environment is the auxiliary state that is used in the quantum teleportation scheme.

Step 1: application of quantum teleportation

We start by recalling the quantum teleportation protocol [1]. Let $\mathcal{H}_T$ be a finite-dimensional Hilbert space. Assume that Alice wants to send a particle, at a (possibly mixed) state x that lives in $\mathcal{H}_T$, to Bob. To do so, they use a certain bipartite quantum state ρ_{AB} such that $\mathcal{H}_T = \mathcal{H}_B$ of which Alice holds the subsystem $\mathcal{H}_A$ and Bob holds the subsystem $\mathcal{H}_B$. Consider the state $x_T \otimes \rho_{AB}$. Alice performs on $\mathcal{H}_T \otimes \mathcal{H}_A$ the von-Neumann measurement corresponding to a certain basis $|\psi^i\rangle, \ldots, |\psi^k\rangle$. If she gets outcome i, Bob performs a certain unitary operation U_i over the system $\mathcal{H}_B$. This sets the system $\mathcal{H}_B$ in state x.

For an Hermitian operator r over $\mathcal{H}_T \otimes \mathcal{H}_A$ let $\delta^i_{TA}(r) = \langle\psi^i|r|\psi^i\rangle$. For an Hermitian operator y over H_B we let $\eta^i(y) = U_i^* y U_i$. Using these notation, we can summarize validity of the protocol with the following equation:

$$\left(\sum_i \delta^i_{TA} \otimes \eta^i_B\right)(x_T \otimes \rho_{AB}) = x_T. \tag{12}$$

For every state x_T, as usual, subscripts of maps correspond to the systems on which they act. Note that the state ρ_{AB} that appears in (12) is the specific (maximally entangled) state that is used in the teleportation protocol. The magic also operates if the original particle was entangled with another particle with corresponding Hilbert space $\mathcal{H}_N$. We summarize this with the following proposition:

Proposition 4. *There exist a state ρ_{AB}, a linear function $\delta = (\delta^1, \ldots, \delta^k)$ from density operators over $\mathcal{H}_T \otimes \mathcal{H}_A$ to Δ_k and trace-preserving completely positive maps η^i over $\mathcal{H}_B$ such that for every density operator Ψ_{NT} over $\mathcal{H}_N \otimes \mathcal{H}_T$ one has*

$$\left(\mathcal{I}_N \otimes \sum_i \left(\delta^i_{TA} \otimes \eta^i_B\right)\right)(\Psi_{NT} \otimes \rho_{AB}) = \Psi_{NT}. \tag{13}$$

Step 2: application of separation theorem

The function $\delta = \left(\delta^1_{TA}, \ldots, \delta^k_{TA}\right)$ that appears in equation (13) corresponds to a strategy in a k-action game over the information structure Ψ_{NT} with the environment ρ_{AB}. We claim that there exists a strategy $\tilde{\delta}_{SA} = \left(\tilde{\delta}^1_{SA}, \ldots, \tilde{\delta}^k_{SA}\right)$ for the information structure Φ_{NS} such that, for every $1 \leqslant i \leqslant k$,

$$\left(\mathcal{I}_{NB} \otimes \tilde{\delta}^i_{SA}\right)(\Phi_{NS} \otimes \rho_{AB}) = \left(\mathcal{I}_{NB} \otimes \delta^i_{TA}\right)(\Psi_{NT} \otimes \rho_{AB}), \tag{14}$$

where ρ_{AB} is the state that appears in proposition 4. Indeed, consider the set C of all k-tuples

$$\left(\left(\mathcal{I}_{NB} \otimes \tilde{\delta}^1_{SA}\right)(\Phi_{NS} \otimes \rho_{AB}), \ldots, \left(\mathcal{I}_{NB} \otimes \tilde{\delta}^k_{SA}\right)(\Phi_{NS} \otimes \rho_{AB})\right)$$

for some strategy $\tilde{\delta}_{SA} = \left(\tilde{\delta}^1_{SA}, \ldots, \tilde{\delta}^k_{SA}\right)$. This is a convex compact set in the linear space of all k-tuples of Hermitian operators over $\mathcal{H}_N \otimes \mathcal{H}_B$.

If $\left(\left(\mathcal{I}_{NB} \otimes \delta^1_{TA}\right)(\Psi_{NT} \otimes \rho_{AB}), \ldots, \left(\mathcal{I}_{NB} \otimes \delta^k_{TA}\right)(\Psi_{NT} \otimes \rho_{AB})\right)$ is outside C then, by the separation theorem for convex sets [8], there exists a hyperplane that separates it from C, i.e. there exist Hermitian operators $(M^1, \ldots, M^k)$ over $\mathcal{H}_N \otimes \mathcal{H}_B$ such that for every strategy $\tilde{\delta}_{SA} = \left(\tilde{\delta}^1_{SA}, \ldots, \tilde{\delta}^k_{SA}\right)$,

$$\sum_i \mathrm{tr}\left(\left(\mathcal{I}_{NB} \otimes \tilde{\delta}^i_{SA}\right)(\Phi \otimes \rho) \cdot M^i\right) < \sum_i \mathrm{tr}\left(\left(\mathcal{I}_{NB} \otimes \delta^i_{TA}\right)(\Psi \otimes \rho) \cdot M^i\right). \tag{15}$$

Consider the game with set of actions $\{M^1, \ldots, M^k\}$ and the environment ρ_{AB}. By (15) and (8), we have

$$\begin{aligned} R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) &< \sum_i \mathrm{tr}\left(\left(\mathcal{I}_{NB} \otimes \delta^i_{SA}\right)(\Psi_{NT} \otimes \rho_{AB}) \cdot M^i\right) \\ &\leqslant R(\Psi_{NT}; \rho_{AB}, M^1, \ldots, M^k), \end{aligned}$$

contradicting the assumption that Φ_{NS} is better than Ψ_{NT}. Thus, there exists a strategy $\tilde{\delta}_{SA} = \left(\tilde{\delta}^1_{SA}, \ldots, \tilde{\delta}^k_{SA}\right)$ satisfying (14).

Step 3: constructing $\mathcal{E}_S$

It follows from (14) that, for every $1 \leqslant i \leqslant k$,

$$\mathcal{I}_N \otimes \tilde{\delta}^i_{SA} \otimes \eta^i_B(\Phi_{NS} \otimes \rho_{AB}) = \mathcal{I}_N \otimes \delta^i_{TA} \otimes \eta^i_B(\Psi_{NT} \otimes \rho_{AB}). \tag{16}$$

From (13) and (16), we get

$$\mathcal{I}_N \otimes \left(\sum_i \left(\tilde{\delta}^i_{SA} \otimes \eta^i_B\right)\right)(\Phi_{NS} \otimes \rho_{AB}) = \Psi_{NT}. \tag{17}$$

Let $\mathcal{E}_S$ be the map defined over the system $\mathcal{H}_S$ by

$$\mathcal{E}_S : x_S \mapsto \sum_i \left(\tilde{\delta}^i_{SA} \otimes \eta^i_B\right)(x_S \otimes \rho_{AB}). \tag{18}$$

This is a trace-preserving completely positive map. From (17) and (18), we get

$$(\mathcal{I}_N \otimes \mathcal{E}_S)(\Phi_{NS}) = \Psi_{NT},$$

as desired.

5. Quantitative measure of information

Recall the useful definition of mutual information of a bipartite state. If ρ_{XY} is a bipartite state, then $I(\rho_{XY})$ is the real number defined by

$$I(\rho_{XY}) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}),$$

where $S(\rho)$ is von-Neumann's entropy of ρ. This is the quantum analogue of mutual information of random variables in classical statistics. The following lemma is an immediate consequence of theorem 2. Its classical analogue is well known.

Lemma 5. *Let $\Phi = \Phi_{NS}$ and $\Psi = \Psi_{NT}$ be two information structures. If Φ is better than Ψ then $I(\Phi) \geqslant I(\Psi)$.*

Proof. By theorem 2, there exists a trace-preserving completely positive map $\mathcal{E}$ such that $\Psi = (\mathcal{I} \otimes \mathcal{E})\Phi$. Since local quantum operation cannot increase mutual information (see, for example, [7] section 11.4.2) it follows that $I(\Phi) \geqslant I(\Psi)$. □

The converse, however, is not true (neither in classical statistics), as the following example shows.

Consider the bipartite state Υ given by

$$\Upsilon = \tfrac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|).$$

The information structure was already discussed in section 1. We consider games in which the observer, who has access to the second particle, has to guess the spin of the first particle along the $\hat{n}$ axis for some $\hat{n} \in \mathbf{R}^3$, with payoff +1 for correct guess and −1 for incorrect guess. The maximal possible payoff is +1 if $\hat{n} = \hat{z}$ and 0 if $\hat{n} = \hat{x}$.

Let $\Upsilon' = (H \otimes H)\Upsilon(H \otimes H)$, where

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

When playing the same games over Υ', the observer maximal payoff is +1 if $\hat{n} = \hat{x}$ and 0 if $\hat{n} = \hat{z}$. Therefore, it follows that even though $I(\Upsilon) = I(\Upsilon') = 1$, neither is better than the other in Blackwell's sense: for some games it is better to play over Υ and for some games it is better to play over Υ'.

6. Games with trivial environment

Theorems 1 and 2 have a similar purpose: to provide an operational interpretation for the specific map. Both use the concept of the game to formulate its operational aspect. The main conceptual difference between them is the additional introduction of the environment. Two notes are in order. First, the class of quantum games which is defined in section 3 includes as a subset games with trivial environment, that is games for which $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 1$. These games are the exact analogue of the classical games that were considered in section 2. Similarly, one can expand the set of classical games to games with classical environment. However, in the classical framework, the concept of games with environment is superfluous for comparison of information structures because of the following claim:

Claim 6. *If (N, S, p) and (N, T, q) are two classical information structures such that the payoff over (N, S, p) is at least as good as the payoff over (N, T, q) for every classical game with trivial environment, then the payoff of (N, S, p) is at least as good as the payoff over (N, T, q) also for games with non-trivial environment.*

The claim follows from theorem 1 and from the fact that if $\mathcal{F}$ is a classical stochastic map then so is $\mathcal{F} \otimes \mathcal{I}$. We do not prove it formally here, in order to avoid the notational encumbrance of classical environment.

In the quantum world, however, there is no analogue for claim 6. There exist information structures Φ and Ψ such that the payoff over Φ is at least as good as the payoff over Ψ for every game with trivial environment, but not for every game with non-trivial environment. This is why we had to explicitly define the order relation 'better' in section 3 using games with the environment. The construction of such a pair Φ and Ψ of information structures, which we now describe, is based on the existence of positive maps which are not completely positive.

Let $\mathcal{H}_N = \mathcal{H}_S = \mathcal{H}_T$ be three n-dimensional Hilbert space. Let $m = n^2$ and $\rho_1, \ldots, \rho_m$ be a set of $n \times n$ density matrices which are linearly independent in the linear space of all $n \times n$ Hermitian matrices. Let $\Phi = \frac{1}{m}(\rho_1 \otimes \rho_1 + \cdots + \rho_m \otimes \rho_m)$ and $\Psi = \frac{1}{m}\left(\rho_1 \otimes \rho_1^t + \cdots + \rho_m \otimes \rho_m^t\right)$, where ρ_j^t is the transpose of ρ_j. There exists only one linear map $\mathcal{E}$ from Hermitian operators to Hermitian operators such that $\Psi = \mathcal{I} \otimes \mathcal{E}(\Phi)$, and this map is given by $\mathcal{E}(H) = H^t$, which is not completely positive. In particular, it follows from theorem 2 that there exists a game (with non-trivial environment) for which Ψ offers higher payoff than Φ. On the other hand, since $\mathcal{E}$ is positive and $\Psi = \mathcal{I} \otimes \mathcal{E}(\Phi)$, it follows from the remark 3 that for every game with trivial environment the payoff under Φ is at least as good as the payoff under Ψ.

Remark 7. One may conjecture that if Φ and Ψ are two structures such that the payoff under Φ is at least as good as the payoff over Ψ for every game with trivial environment, then there exists a positive (but not necessarily completely positive) trace-preserving map $\mathcal{E}$ such that $\Psi = (\mathcal{I} \otimes \mathcal{E})\Phi$. However, I do not know whether this is true.

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